

Differential and Integral Equation

Q-1 Solve the following integral equation

$$u(x) = 1 + \lambda \int_0^1 (x+t) u(t) dt$$

by the method of successive approximation upto third order.

Solⁿ — Here the given integral equation is

$$u(x) = 1 + \lambda \int_0^1 (x+t) u(t) dt \quad \text{--- (1)}$$

Let us assume

$$u_0(x) = 1 \quad \text{--- (2)}$$

then the n th approximation is given by

$$u_n(x) = 1 + \lambda \int_0^1 (x+t) u_{n-1}(t) dt \quad \text{--- (3)}$$

$$\Rightarrow u_1(x) = 1 + \lambda \int_0^1 (x+t) u_0(t) dt$$

$$= 1 + \lambda \int_0^1 (x+t) \cdot 1 \cdot dt$$

$$= 1 + \lambda \left[xt + \frac{1}{2} t^2 \right]_0^1$$

$$= 1 + \lambda \left(x + \frac{1}{2} \right)$$

Also from equation (3), we get

$$u_2(x) = 1 + \lambda \int_0^1 (x+t) u_1(t) dt$$

$$= 1 + \lambda \int_0^1 (x+t) \left\{ 1 + \lambda \left(t + \frac{1}{2} \right) \right\} dt$$

$$\begin{aligned}
 &= 1 + \lambda \int_0^1 \left[x \left(1 + \frac{\lambda}{2} \right) + t \left[t + \frac{\lambda}{2} + \lambda x \right] + \lambda t^2 \right] dt \\
 &= 1 + \lambda \left[x \left(1 + \frac{\lambda}{2} \right) t + \frac{t^2}{2} \left(1 + \frac{\lambda}{2} + \lambda x \right) + \frac{\lambda t^3}{3} \right]_0^1 \\
 &= 1 + \lambda \left(x + \frac{1}{2} \right) + \lambda^2 \left[x + \frac{7}{12} \right] \quad \text{--- (4)}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 u_3(x) &= 1 + \lambda \int_0^1 (x+t) u_2(t) dt \\
 &= 1 + \lambda \int_0^1 (x+t) \left\{ 1 + \lambda \left(t + \frac{1}{2} \right) + \lambda^2 \left(t + \frac{7}{12} \right) \right\} dt \\
 &= 1 + \lambda \int_0^1 (x+t) \left\{ \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} \right) + \lambda t (1 + \lambda) \right\} dt \\
 &= 1 + \lambda \left[\left\{ x \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} \right) + \frac{t^2}{2} \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} + x\lambda + x\lambda^2 \right) + \frac{1}{3} \lambda t^3 (1 + \lambda) \right\} \right]_0^1 \\
 &= 1 + \lambda x \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} \right) + \frac{\lambda}{2} \left(1 + \frac{\lambda}{2} + \frac{7\lambda^2}{12} + x\lambda + x\lambda^2 \right) + \frac{1}{3} \lambda^2 (1 + \lambda) \\
 &= 1 + \lambda \left(x + \frac{1}{2} \right) + \lambda^2 \left(x + \frac{7}{12} \right) + \lambda^3 \left(\frac{13}{12} x + \frac{5}{8} \right)
 \end{aligned}$$

